647510 **MSC3MatC302x** **Seat No :________**

M.Sc.(Maths.) Semester - 3 (*CBCS*) Examination
Oct/Nov-2021 (NEW COURSE)
Functional Analysis (Core (New))

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1 (a) Answer the Following. **4 Marks**

(1) Consider the vector space X of all complex sequences with the usual pointwise addition and scalar multiplication. Define $p: X \to [0, \infty)$ by $p\big((x_n)\big) = |x_2|$. Show that p is not a norm on X.

(2) Let $T: \ell^2 \to \ell^2$ be defined as $T\big((x_1, x_2, \ldots)\big) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots)$. Compute the norm of T.

Q.1 (b) Answer any two question out of three. **10 Marks**

(1) Let Y be a closed subspace of a normed space $(X, \|\cdot\|)$. Define $\|x + Y\|_0 = \inf\{\|x + y\| : y \in Y\}$. Show that X/Y is a normed space with the norm $\|\cdot\|_0$.

(2) Let $P[0,1]$ be the normed space of all polynomials on $[0,1]$ with the norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Let $T: P[0,1] \to P[0,1]$ be defined by $T(x) = x'$. Show that T is a linear operator but T is not continuous.

(3) If X is a normed spaces, then show that its dual X' is a Banach space.

Q.2 (a) Answer the Following. **4 Marks**

(1) Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on $\mathbb{C}^2$ by $\|(z_1, z_2)\|_1 = |z_1| + |z_2|$ and $\|(z_1, z_2)\|_2 = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}$. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

(2) Let (x_n) be a sequence in a normed space X, and let $x \in X$. If $\|x_n - x\| \to 0$ as $n \to \infty$, then show that the sequence (x_n) converges weakly.

Q.2 (b) Answer any two question out of three. **10 Marks**

(1) Let Y and Z be subspaces of a normed space X, and suppose that Y is closed and is a proper subset of Z. Then show that for every real number $\theta \in (0,1)$ there is a $z \in Z$ such that $\|z\| = 1$ and $\|z - y\| \geq \theta$ for all $y \in Y$.

(2) If Y is a finite dimensional subspace of a normed space X, then show that Y is complete with the norm on X.

(3) Let X be a finite dimensional normed space, and let M be a subset of X. Show that M is compact if and only if M is closed and bounded.

Q.3 (a) Answer the Following. **4 Marks**

(1) Let x be an element of a normed space X. Define a linear operator $J_x: X' \to \mathbb{C}$ by $J_x(f) = f(x)$ for all $f \in X'$. Show that $\|J_x\| \geq \|x\|$.

(2) Let X be a Banach space, f be a nonzero continuous linear functional on X, and let B be the open unit ball of X. Use open mapping theorem to show that $f(B)$ is an open set.

Q.3 (b) Answer any two question out of three. **10 Marks**

(1) Let f be a bounded linear functional on a subspace Z of a normed space X. Show that there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and $\|\tilde{f}\| = \|f\|$.

(2) Let X and Y be Banach spaces and $T: X \to Y$ a closed linear operator. Show that the operator T is bounded.

(3) Let $T: X \to Y$ be a bounded linear operator, where X and Y are Banach spaces. If T is bijective, then show that there are positive real numbers a and b such that $a\|x\| \leq \|Tx\| \leq b\,\|x\|$ for all $x \in X$.

Q.4 (a) **Answer the Following.** **4 Marks**

(1) Let X be an inner product space. If S is a nonempty subset of X, then show that $S^{\perp}$ is a subspace of X.

(2) Let H be a Hilbert space, and let $P: H \rightarrow H$ be a bounded linear operator satisfying $P^2 = P$. Show that P is a closed map.

Q.4 (b) **Answer any two question out of three.** **10 Marks**

(1) Let H be a Hilbert space, $\{e_\alpha\}_{\alpha\in\Lambda}$ be a total orthonormal set, and let $x \in H$. Show that the set $\{\alpha \in \Lambda: \langle x, e_\alpha\rangle \neq 0\}$ is a countable set.

(2) If f is bounded linear functional on a Hilbert space H, then show that there is unique $z \in H$ such that $f(x) = \langle x, z\rangle$ for all $x \in H$ and $\|z\| = \|f\|$.

(3) Let $1 \leq p \leq \infty$. If $p \neq 2$, then show that $(\ell^p, \|\cdot\|_p)$ is not a Hilbert space.

Q.5 (a) **Answer the Following.** **4 Marks**

(1) Let X and Y be inner product spaces, and let $T: X \rightarrow Y$ be a bounded linear operator. Show that $T = 0$ if and only if $\langle Tx, y\rangle = 0$ for all $x \in X$ and $y \in Y$.

(2) If T is a self-adjoint operator on a Hilbert space H, then show that $\langle Tx, x\rangle$ is real for all $x \in H$.

Q.5 (b) **Answer any two question out of three.** **10 Marks**

(1) If two bounded self-adjoint linear operators S and T on a Hilbert space H are positive and commute, then show that their product ST is positive.

(2) Let $\mathcal{L}$ be the collection of all self-adjoint operators on a Hilbert space H. Show that $\mathcal{L}$ is closed in $B(H, H)$. Is $\mathcal{L}$ a subspace of $B(H, H)$? Why?

(3) Let T be a bounded linear operator on a Hilbert space H. Show that there exist unique self-adjoint operators T_1 and T_2 on H such that $T = T_1 + iT_2$.

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